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2006 J. Phys. A: Math. Gen. 39 10263

(http://iopscience.iop.org/0305-4470/39/33/001)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 10263–10275 doi:10.1088/0305-4470/39/33/001

Spanning trees on lattices and integral identities

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Received 26 April 2006, in final form 8 July 2006
Published 2 August 2006
Online at stacks.iop.org/JPhysA/39/10263

Abstract
For a lattice Λ with n vertices and dimension d equal to or higher than 2, the number of spanning trees $N_{\mathrm{ST}}(\Lambda)$ increases asymptotically as $\exp(nz_\Lambda)$ in the thermodynamic limit. We present exact integral expressions for the asymptotic growth constant z_Λ for spanning trees on several lattices. By taking different unit cells in the calculation, many integral identities can be obtained. We also give $z_{\Lambda(p)}$ on the homeomorphic expansion of k-regular lattices with p vertices inserted on each edge.

PACS numbers: 02.10.Ox, 05.20.−y

1. Introduction

The enumeration of the number of spanning trees $N_{\mathrm{ST}}(G)$ on the graph G was first considered by Kirchhoff in the analysis of electric circuits [1]. It is a problem of fundamental interest in mathematics [2–5] and physics [6, 7]. The number of spanning trees is closely related to the partition function of the q-state Potts model in statistical mechanics [8, 9]. There are several ways to calculate $N_{\mathrm{ST}}(G)$, including as a determinant of the Laplacian matrix of G and as a special case of the Tutte polynomial of G [2]. Some recent studies on the enumeration of spanning trees and the calculation of their asymptotic growth constants were carried out in [10–12]. In this paper we shall present exact integrals for the asymptotic growth constant for spanning trees on five Archimedean lattices (defined in the next section) and the diamond structure. The hypercubic lattices with edges connecting both nearest-neighbour and next-nearest-neighbour vertices will also be considered. We shall show how integral identities can be obtained with different choices of unit cells in the calculation. We shall also derive the asymptotic growth constant for the homeomorphic expansion of a regular lattice.

2. Background and method

We briefly recall some definitions and background on spanning trees and the calculation method that we use [2, 13]. Let $G = (V, E)$ denote a connected graph (without loops) with vertex (site) and edge (bond) sets V and E. Let $n = v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. A spanning subgraph G' is a subgraph of G with $v(G') = |V|$, and a tree is a connected subgraph with no circuits. A spanning tree is a spanning subgraph of G that is a tree and hence $e(G') = n - 1$. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. Two vertices are adjacent if they are connected by an edge. In general, one can associate an edge weight x_{ij} with each edge connecting adjacent vertices v_i and v_j (see, for example [10]). For simplicity, all edge weights are set to 1 throughout this paper. The adjacency matrix $A(G)$ of G is the $n \times n$ matrix with elements $A(G)_{ij} = 1$ if v_i and v_j are adjacent and zero otherwise. The Laplacian matrix $Q(G)$ is the $n \times n$ matrix with element $Q(G)_{ij} = k_i\delta_{ij} - A(G)_{ij}$. One of the eigenvalues of $Q(G)$ is always zero; let us denote the rest as $\lambda(G)_i$, $1 \leqslant i \leqslant n-1$. A basic theorem is that $N_{\mathrm{ST}}(G) = (1/n)\prod_{i=1}^{n-1}\lambda(G)_i$ [2]. For d-dimensional lattices Λ with $d \geqslant 2$ in the thermodynamic limit, $N_{\mathrm{ST}}(\Lambda)$ increases exponentially with n as $n \to \infty$; that is, there exists a constant z_Λ such that $N_{\mathrm{ST}}(\Lambda) \sim \exp(nz_\Lambda)$ as $n \to \infty$. The constant describing this exponential growth is thus given by [4, 5]

$$z_\Lambda = \lim_{n\to\infty} n^{-1} \ln N_{\mathrm{ST}}(\Lambda), \tag{1}$$

where Λ, when used as a subscript in this manner, implicitly refers to the thermodynamic limit of the lattice Λ. If each pair of neighbouring vertices is connected by p-fold edges instead of just one edge, the number of spanning trees should be multiplied by the factor p^{n-1} and the resultant asymptotic growth constant increases by the factor $\ln p$. Similarly, it is easy to generalize to the case with edges replaced by different numbers of multiple edges. Henceforth, we only consider simple graphs without multiple edges. A regular d-dimensional lattice is comprised of repeated unit cells, each containing ν vertices. Define $a(\tilde{n}, \tilde{n}')$ as the $\nu \times \nu$ matrix describing the adjacency of the vertices of the unit cells $\tilde{n}$ and $\tilde{n}'$, the elements of which are given by $a(\tilde{n}, \tilde{n}')_{ij} = 1$ if $v_i \in \tilde{n}$ is adjacent to $v_j \in \tilde{n}'$ and zero otherwise. Although the number of spanning trees $N_{\mathrm{ST}}(\Lambda)$ depends on the boundary conditions imposed as shown in [10], the asymptotic growth constant z_Λ is not sensitive to them. We consider a given lattice having periodic boundary conditions, such that each of its vertices has the same degree k, i.e. k-regular. Using the resultant translational symmetry for the spanning trees, we have $a(\tilde{n}, \tilde{n}') = a(\tilde{n} - \tilde{n}')$, and we can therefore write $a(\tilde{n}) = a(\tilde{n}_1, \dots, \tilde{n}_d)$. By the method derived in [11], $N_{\mathrm{ST}}(\Lambda)$ and z_Λ can be calculated in terms of a matrix M which is determined by $a(\tilde{n})$. For a d-dimensional lattice which is k-regular, define

$$M(\theta_1, \dots, \theta_d) = k \cdot 1 - \sum_{\tilde{n}} a(\tilde{n})\, \mathrm{e}^{\mathrm{i}\tilde{n}\cdot\Theta}, \tag{2}$$

where 1 is the unit matrix and Θ stands for the d-dimensional vector $(\theta_1, \dots, \theta_d)$. Then [4, 11]

$$z_\Lambda = \frac{1}{\nu}\int_{-\pi}^{\pi}\left[\prod_{j=1}^{d}\frac{\mathrm{d}\theta_j}{2\pi}\right]\ln[D(\theta_1, \dots, \theta_d)], \tag{3}$$

where $D(\theta_1, \dots, \theta_d) = \det(M(\theta_1, \dots, \theta_d))$ is the determinant of the matrix.

An Archimedean lattice is a uniform tiling of the plane by regular polygons in which all vertices are equivalent [14, 15]. Such a lattice can be defined by the ordered sequence of polygons that one traverses in making a complete circuit around the local neighbourhood of any vertex. This is indicated by the notation $\Lambda = \left(\prod_i p_i^{a_i}\right)$, meaning that in this circuit, the regular p_i-sided polygon occurs contiguously a_i times. There are altogether 11 Archimedean lattices and all 11 are k-regular [14].

The number of spanning trees is the same for a planar graph G and its dual G^*, and the number of the vertices of G^* is given by the Euler relation $n^* = |E| - n + 1$. Because

$|E|/n = k/2$ for a k-regular graph G_k, the asymptotic growth constants of G_k and G_k^* satisfy [11]

$$z_{G_k^*} = \frac{z_{G_k}}{k/2 - 1}. \tag{4}$$

Consequently, $z_{G_3^*} = 2z_{G_3}$, $z_{G_4^*} = z_{G_4}$, $z_{G_5^*} = 2z_{G_5}/3$ and $z_{G_6^*} = z_{G_6}/2$.

For a k-regular graph G_k, a general upper bound is $z_{G_k} \leqslant \ln k$. A stronger upper bound for a k-regular graph G_k with $k \geqslant 3$ can be obtained from the bound [16, 17]

$$N_{\mathrm{ST}}(G_k) \leqslant \left(\frac{2\ln n}{nk\ln k}\right)(b_k)^n, \tag{5}$$

where

$$b_k = \frac{(k-1)^{k-1}}{[k(k-2)]^{\frac{k}{2}-1}}. \tag{6}$$

With equation (1), this then yields [11]

$$z_{G_k} \leqslant \ln(b_k). \tag{7}$$

Two graphs G and G' are homeomorphic to each other if one of them, say G', can be obtained from the other, G, by successive insertions of degree-2 vertices on edges of G [13]. This process is called homeomorphic expansion. We shall denote a lattice Λ with p vertices inserted on each edge as $\Lambda(p)$.

3. Asymptotic growth constants

The asymptotic growth constants z_Λ for many $d \geqslant 2$ lattices have been considered by several authors [6, 7, 10–12]. The known expressions for certain two-dimensional lattices are as follows: $z_{sq} = 4C/\pi$ for the square lattice, $z_{tri} = (\ln 3)/2 + (6/\pi)\mathrm{Ti}_2(1/\sqrt{3})$ for the triangular lattice [18, 19] and $z_{(4\cdot8\cdot8)} = C/\pi + (\ln(\sqrt{2}-1))/2 + (\mathrm{Ti}_2(3+2\sqrt{2}))/\pi$ [12], where $C = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2} = 0.915\,965\,594\,177\ldots$ is the Catalan constant and Ti_2 is the inverse tangent integral function [20] given in equation (32). While $z_{hc} = z_{tri}/2$ for the honeycomb lattice due to the duality given in equation (4), it is non-trivial to have the relations $z_{kag} = (z_{tri} + \ln 6)/3$ for the Kagomé (equivalently $(3\cdot6\cdot3\cdot6)$) lattice and $z_{(3\cdot12\cdot12)} = (z_{tri} + \ln(15))/6$ given in [11]. In addition to the six lattices mentioned above, there are five more Archimedean lattices that have not been considered before. Namely, $(3^3\cdot4^2)$, $(3^2\cdot4\cdot3\cdot4)$, $(3^4\cdot6)$, $(3\cdot4\cdot6\cdot4)$ and $(4\cdot6\cdot12)$ which will be studied in this section. Most of the calculations done so far have been performed for lattices with edges connecting nearest-neighbour vertices. The consideration can be extended to include edges connecting farther vertices. We shall take the d-dimensional hypercubic lattice with edges connecting both nearest-neighbour and next-nearest-neighbour vertices as an illustration.

3.1. $(3^3\cdot4^2)$ *lattice*

The $(3^3\cdot4^2)$ lattice can be constructed by starting with the square lattice and adding a diagonal edge connecting the vertices in, say, the upper left to the lower right corners of each square in every other row as shown in figure 1(*a*). The simplest unit cell for this lattice contains two vertices $\nu_{(3^3\cdot4^2)} = 2$ as labelled in figure 1(*a*). We have

$$M_{(3^3\cdot4^2)}(\theta_1,\theta_2) = \begin{pmatrix} 5-2\cos\theta_1 & -1-\mathrm{e}^{\mathrm{i}\theta_1}-\mathrm{e}^{\mathrm{i}\theta_2} \\ -1-\mathrm{e}^{-\mathrm{i}\theta_1}-\mathrm{e}^{-\mathrm{i}\theta_2} & 5-2\cos\theta_1 \end{pmatrix}. \tag{8}$$

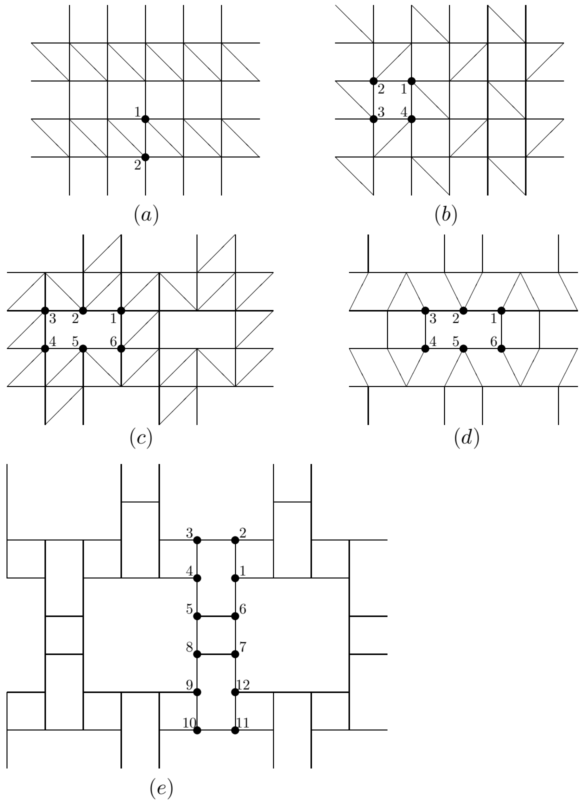

Figure 1. (*a*) The $(3^3 \cdot 4^2)$ lattice. (*b*) The $(3^2 \cdot 4 \cdot 3 \cdot 4)$ lattice. (*c*) The $(3^4 \cdot 6)$ lattice. (*d*) The $(3 \cdot 4 \cdot 6 \cdot 4)$ lattice. (*e*) The $(4 \cdot 6 \cdot 12)$ lattice. Vertices within a unit cell are labelled.

The determinant can be calculated as

$$D_{(3^3 \cdot 4^2)}(\theta_1, \theta_2) = 22 - 22\cos\theta_1 + 4\cos^2\theta_1 - 2\cos\theta_2 - 2\cos(\theta_1 - \theta_2) \tag{9}$$

such that

$$z_{(3^3 \cdot 4^2)} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{(3^3 \cdot 4^2)}(\theta_1, \theta_2)] = \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\ln[11 - 11\cos\theta + 2\cos^2\theta + \sqrt{(1-\cos\theta)(7-2\cos\theta)(17-13\cos\theta+2\cos^2\theta)}] = 1.406\,925\,8315\ldots, \tag{10}$$

where one integration is carried out followed by a numerical evaluation of the remaining integral.

3.2. $(3^2 \cdot 4 \cdot 3 \cdot 4)$ *lattice*

The $(3^2 \cdot 4 \cdot 3 \cdot 4)$ lattice is shown in figure 1(*b*) which has the structure of a net of squares. Taking each square as a unit cell with $\nu_{(3^2 \cdot 4 \cdot 3 \cdot 4)} = 4$, we have

$$M_{(3^2 \cdot 4 \cdot 3 \cdot 4)}(\theta_1, \theta_2) = \begin{pmatrix} 5 & -1 - e^{i\theta_1} & -e^{i\theta_1} & -1 - e^{i\theta_2} \\ -1 - e^{-i\theta_1} & 5 & -1 - e^{i\theta_2} & -e^{i\theta_2} \\ -e^{-i\theta_1} & -1 - e^{-i\theta_2} & 5 & -1 - e^{-i\theta_1} \\ -1 - e^{-i\theta_2} & -e^{-i\theta_2} & -1 - e^{i\theta_1} & 5 \end{pmatrix}. \tag{11}$$

The determinant can be calculated as

$$D_{(3^2\cdot4\cdot3\cdot4)}(\theta_1,\theta_2) = 336 - 144(\cos\theta_1+\cos\theta_2) + 4(\cos^2\theta_1+\cos^2\theta_2) - 56\cos\theta_1\cos\theta_2 \tag{12}$$

such that

$$\begin{aligned} z_{(3^2\cdot4\cdot3\cdot4)} &= \frac{1}{4}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{(3^2\cdot4\cdot3\cdot4)}(\theta_1,\theta_2)] = \ln 2 + \frac{1}{4}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi} \\ &\quad\times\ln\left[6+4\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)-\cos^2\left(\frac{\theta_1}{2}\right)-\cos^2\left(\frac{\theta_2}{2}\right)\right] \\ &\quad+\frac{1}{4}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln\left[6-4\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)-\cos^2\left(\frac{\theta_1}{2}\right)-\cos^2\left(\frac{\theta_2}{2}\right)\right] \\ &= 1.410\,855\,6457\ldots. \end{aligned} \tag{13}$$

3.3. $(3^4\cdot6)$ *lattice*

The $(3^4\cdot6)$ lattice is shown in figure 1(*c*) which has the structure of a net of hexagons. Taking each hexagon as a unit cell with $\nu_{(3^4\cdot6)}=6$, we have

$$M_{(3^4\cdot6)}(\theta_1,\theta_2) = \begin{pmatrix} 5 & -1 & -e^{i\theta_1} & -e^{i(\theta_1+\theta_2)} & -e^{i(\theta_1+\theta_2)} & -1 \\ -1 & 5 & -1 & -e^{i(\theta_1+\theta_2)} & -e^{i\theta_2} & -e^{i\theta_2} \\ -e^{-i\theta_1} & -1 & 5 & -1 & -e^{i\theta_2} & -e^{-i\theta_1} \\ -e^{-i(\theta_1+\theta_2)} & -e^{-i(\theta_1+\theta_2)} & -1 & 5 & -1 & -e^{-i\theta_1} \\ -e^{-i(\theta_1+\theta_2)} & -e^{-i\theta_2} & -e^{-i\theta_2} & -1 & 5 & -1 \\ -1 & -e^{-i\theta_2} & -e^{i\theta_1} & -e^{i\theta_1} & -1 & 5 \end{pmatrix} \tag{14}$$

and the determinant $D_{(3^4\cdot6)}(\theta_1,\theta_2)$ is given by $f(5032, 1624, 56)$, where we define the function

$$\begin{aligned} f(a,b,c) &= a - b[\cos\theta_1+\cos\theta_2+\cos(\theta_1-\theta_2)] - c[\cos\theta_1\cos\theta_2 \\ &\quad+(\cos\theta_1+\cos\theta_2)\cos(\theta_1-\theta_2)] + 8\cos\theta_1\cos\theta_2\cos(\theta_1-\theta_2). \end{aligned} \tag{15}$$

Thus,

$$z_{(3^4\cdot6)} = \frac{1}{6}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[f(5032,1624,56)] = 1.392\,023\,5634\ldots. \tag{16}$$

This result applies to both enantiomorphic forms of the $(3^4\cdot6)$ lattice.

3.4. $(3\cdot4\cdot6\cdot4)$ *lattice*

The $(3\cdot4\cdot6\cdot4)$ lattice is shown in figure 1(*d*) which also has the structure of a net of hexagons. Taking each hexagon as a unit cell with $\nu_{(3\cdot4\cdot6\cdot4)}=6$, we have

$$M_{(3\cdot4\cdot6\cdot4)}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & -e^{i\theta_1} & 0 & -e^{i(\theta_1+\theta_2)} & -1 \\ -1 & 4 & -1 & -e^{i(\theta_1+\theta_2)} & 0 & -e^{i\theta_2} \\ -e^{-i\theta_1} & -1 & 4 & -1 & -e^{i\theta_2} & 0 \\ 0 & -e^{-i(\theta_1+\theta_2)} & -1 & 4 & -1 & -e^{-i\theta_1} \\ -e^{-i(\theta_1+\theta_2)} & 0 & -e^{-i\theta_2} & -1 & 4 & -1 \\ -1 & -e^{-i\theta_2} & 0 & -e^{i\theta_1} & -1 & 4 \end{pmatrix}. \tag{17}$$

The determinant $D_{(3\cdot4\cdot6\cdot4)}(\theta_1,\theta_2)$ is given by $f(1144, 376, 8)$ such that

$$z_{(3\cdot4\cdot6\cdot4)} = \frac{1}{6}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[f(1144,376,8)] = 1.144\,801\,1236\ldots. \tag{18}$$

3.5. (4 · 6 · 12) *lattice*

The (4 · 6 · 12) lattice is shown in figure 1(*e*), where every two neighbouring hexagons are taken as a unit cell with $\nu_{(4\cdot 6\cdot 12)} = 12$. We have

$$M_{(4\cdot6\cdot12)}(\theta_1,\theta_2) = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -e^{i\theta_1} & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -e^{i\theta_1} & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{i\theta_2} \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -e^{i\theta_2} & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -e^{-i\theta_1} & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ -e^{-i\theta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -e^{-i\theta_2} & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -e^{-i\theta_2} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}. \tag{19}$$

The determinant $D_{(4\cdot6\cdot12)}(\theta_1,\theta_2)$ is given by $f(13432, 4344, 136)$ such that

$$z_{(4\cdot6\cdot12)} = \frac{1}{12}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[f(13432, 4344, 136)] = 0.777\,795\,50613\ldots. \tag{20}$$

3.6. d-dimensional hypercubic lattice with additional edges connecting next-nearest-neighbour vertices

The d-dimensional hypercubic lattice, denoted as $c(d)$, with edges connecting nearest-neighbour vertices has been studied before in [10, 11]. Now consider the hypercubic lattice with additional edges connecting next-nearest-neighbour vertices, denoted as $c(d), nnn$. The one-dimensional circuit graph with additional edges connecting next-nearest-neighbour vertices corresponds to a strip of the triangular lattice with width 2. We have

$$M_{c(1),nnn}(\theta) = \begin{pmatrix} 4-2\cos\theta & -1-e^{i\theta} \\ -1-e^{-i\theta} & 4-2\cos\theta \end{pmatrix}. \tag{21}$$

The determinant $D_{c(1),nnn}(\theta)$ is equal to $2(1-\cos\theta)(7-2\cos\theta)$ such that

$$z_{c(1),nnn} = (1/2)\ln[(7+3\sqrt{5})/2] = 0.962\,423\,65011\ldots, \tag{22}$$

which agrees with [11, 21]. For $d \geqslant 2$, the coordination number is given by $k_{c(d),nnn} = 2d + 4\binom{d}{2} = 2d^2$. We have the general expression

$$z_{c(d),nnn} = \int_{-\pi}^{\pi}\left[\prod_{i=1}^{d}\frac{d\theta_i}{2\pi}\right]\ln\left(2d^2 - 2\sum_{i=1}^{d}\cos\theta_i - 2\sum_{i\neq j}\cos\theta_i\cos\theta_j\right) \quad \text{for} \quad d \geqslant 2. \tag{23}$$

The square lattice with additional edges connecting next-nearest-neighbour vertices corresponds to $c(d=2), nnn$ such that

$$\begin{aligned} z_{sq,nnn} &= \int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[8-2\cos\theta_1-2\cos\theta_2-4\cos\theta_1\cos\theta_2] \\ &= \int_0^{\pi}\frac{d\theta}{\pi}\ln[4-\cos\theta+\sqrt{3(1-\cos\theta)(5+\cos\theta)}]. \end{aligned} \tag{24}$$

An exact closed-form expression for this integral can be derived. We begin by recasting the integral in the equivalent form.

$$z_{\mathrm{sq},nnn} = \ln 3 + \frac{4}{\pi}\int_0^{\pi/2} \mathrm{d}\phi \ln[\sin\phi + \sqrt{1-(1/3)\sin^2\phi}]$$
$$= \ln 3 + 4I(1/\sqrt{3}), \tag{25}$$

where

$$I(a) = \frac{1}{\pi}\int_0^{\pi/2} \mathrm{d}\phi \ln(\sin\phi + \sqrt{1-a^2\sin^2\phi}). \tag{26}$$

In equation (26), without loss of generality, we take a to be non-negative. We will give a general result for $I(a)$ with $0 \leqslant a < 1$ and then specialize to our case $a = 1/\sqrt{3}$. First, we note that $I(1) = C/\pi$, where C is the Catalan constant. Next, taking the derivative with respect to a and doing the integral over ϕ in equation (26), we get

$$I'(a) = \frac{a/2 - (2/\pi)\tanh^{-1} a}{1+a^2}. \tag{27}$$

To calculate $I(a)$, we then use $I(a) - I(0) = \int_0^a I'(x)\,\mathrm{d}x$ and observe that

$$I(0) = \frac{1}{\pi}\int_0^{\pi/2} \mathrm{d}\phi \ln(\sin\phi + 1) = -\frac{\ln 2}{2} + \frac{2C}{\pi}. \tag{28}$$

We also make use of the integrals

$$\int_0^a \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\ln(1+a^2) \tag{29}$$

and

$$\int_0^a \frac{\tanh^{-1} x}{1+x^2}\,\mathrm{d}x = \tan^{-1} a \tanh^{-1} a + \frac{C}{2} + \frac{\pi}{8}\ln\left(\frac{1+a}{1-a}\right) - \frac{1}{2}\mathrm{Ti}_2\left(\frac{1+a}{1-a}\right) \tag{30}$$

to obtain

$$I(a) = \frac{C}{\pi} + \frac{1}{4}\ln\left(\frac{(1+a^2)(1-a)}{4(1+a)}\right) - \frac{2}{\pi}\tan^{-1} a \tanh^{-1} a + \frac{1}{\pi}\mathrm{Ti}_2\left(\frac{1+a}{1-a}\right). \tag{31}$$

Here $\mathrm{Ti}_2(x)$ is the inverse tangent integral [20],

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\tan^{-1} y}{y}\,\mathrm{d}y$$
$$= x[{}_3F_2([1, 1/2, 1/2], [3/2, 3/2], -x^2)], \tag{32}$$

where ${}_pF_q$ is the generalized hypergeometric function,

$${}_pF_q([a_1,\ldots,a_p],[b_1,\ldots,b_q],x) = \sum_{k=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{r=1}^{q}(b_r)_k}\right)\frac{x^k}{k!} \tag{33}$$

with notation $(c)_k = \Gamma(c+k)/\Gamma(c)$. Evaluating our result (31) for $I(a)$ at $a = 1/\sqrt{3}$ and substituting into equation (25), we obtain the exact, closed-form expression

$$z_{\mathrm{sq},nnn} = \frac{4C}{\pi} + \ln(2-\sqrt{3}) - \frac{4}{3}\tanh^{-1}\frac{1}{\sqrt{3}} + \frac{4}{\pi}\mathrm{Ti}_2(2+\sqrt{3}) = 1.943\,739\,3602\ldots. \tag{34}$$

We note that z_{sq} and z_{tri} for the square and triangular lattices, respectively, can be expressed in terms of the Ti_2 function [18, 19]. The numerical evaluations for $3 \leqslant d \leqslant 5$ are given by

$$z_{c(3),nnn} = 2.841\,787\,5363\ldots$$
$$z_{c(4),nnn} = 3.442\,108\ldots \tag{35}$$
$$z_{c(5),nnn} = 3.898\,251\ldots.$$

One observation for the $c(d), nnn$ lattice is that $z_{c(d),nnn} \to \ln(2d^2)$ for $d \to \infty$. Similar behaviour was found for other d-dimensional lattices [11]. In general, for a k-regular d-dimensional lattice $\Lambda_{k(d)}$, the coordination number $k(d)$ is an increasing function of d such that $z_{\Lambda_{k(d)}}$ approaches $\ln(k(d))$ from below in the large d limit.

4. Integral identities

As the method reviewed in section 2 is general, there is a freedom to specify the unit cell for the calculation. Many integral identities can be obtained with different choices of unit cells in the calculation of the spanning trees [11].

4.1. Bcc lattice

Consider the body-centred cubic (bcc) lattice with coordination number $k_{\mathrm{bcc}} = 8$ as the first example. Although the conventional bcc cell contains two vertices, the primitive cell contains only one vertex. In terms of the unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ of the Cartesian coordinate, the primitive translation vectors of the bcc lattice are $(\hat{x}+\hat{y}-\hat{z})/2$, $(\hat{x}-\hat{y}+\hat{z})/2$ and $(-\hat{x}+\hat{y}+\hat{z})/2$. Hence,

$$z_{\mathrm{bcc}} = 3\ln 2 + \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_3}{2\pi}\ln[F_{\mathrm{bcc}}(\theta_1,\theta_2,\theta_3)], \tag{36}$$

where

$$F_{\mathrm{bcc}}(\theta_1,\theta_2,\theta_3) = 1 - \tfrac{1}{4}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \cos(\theta_1+\theta_2+\theta_3)). \tag{37}$$

Comparing z_{bcc} with the expressions given in equation (5.2.7) of [11] and in equation (16) of [12], we find

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_3}{2\pi}\ln[F_{\mathrm{bcc}}(\theta_1,\theta_2,\theta_3)] &= -\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{(2\ell)!}{2^{2\ell}(\ell!)^2}\right)^3 \\ &= -\frac{{}_5F_4([1,1,3/2,3/2,3/2],[2,2,2,2],1)}{2^4}.\end{aligned} \tag{38}$$

4.2. Fcc lattice

Next, consider the face-centred-cubic (fcc) lattice with coordination number $k_{\mathrm{fcc}} = 12$. The conventional fcc cell contains four vertices, but the primitive cell again contains only one vertex. The primitive translation vectors of the fcc lattice are $(\hat{x}+\hat{y})/2$, $(\hat{y}+\hat{z})/2$ and $(\hat{z}+\hat{x})/2$. Hence,

$$z_{\mathrm{fcc}} = \ln(12) + \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_3}{2\pi}\ln[F_{\mathrm{fcc}}(\theta_1,\theta_2,\theta_3)], \tag{39}$$

where

$$F_{\mathrm{fcc}}(\theta_1,\theta_2,\theta_3) = 1 - \tfrac{1}{6}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \cos(\theta_1-\theta_2) + \cos(\theta_2-\theta_3) + \cos(\theta_3-\theta_1)). \tag{40}$$

The equality of this triple integral and the triple integral given in equations (18) and (19) of [12] is therefore established.

4.3. Diamond structure

The diamond structure can be viewed as two fcc structures displaced from each other by one quarter of a cubic diagonal. Each vertex has four nearest neighbours, $k_{\rm diamond} = 4$, at the corners of a tetrahedron. The conventional unit cube contains eight vertices, so that $\nu_{\rm diamond} = 8$, located at $(0, 0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right)$, $\left(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right)$ and $\left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right)$. We have

$$M_{\rm diamond} = \begin{pmatrix} 4 & 0 & 0 & 0 & -1 & -{\rm e}^{-{\rm i}(\theta_1+\theta_2)} & -{\rm e}^{-{\rm i}(\theta_1+\theta_3)} & -{\rm e}^{-{\rm i}(\theta_2+\theta_3)} \\ 0 & 4 & 0 & 0 & -1 & -1 & -{\rm e}^{-{\rm i}\theta_3} & -{\rm e}^{-{\rm i}\theta_3} \\ 0 & 0 & 4 & 0 & -1 & -{\rm e}^{-{\rm i}\theta_2} & -1 & -{\rm e}^{-{\rm i}\theta_2} \\ 0 & 0 & 0 & 4 & -1 & -{\rm e}^{-{\rm i}\theta_1} & -{\rm e}^{-{\rm i}\theta_1} & -1 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 & 0 \\ -{\rm e}^{{\rm i}(\theta_1+\theta_2)} & -1 & -{\rm e}^{{\rm i}\theta_2} & -{\rm e}^{{\rm i}\theta_1} & 0 & 4 & 0 & 0 \\ -{\rm e}^{{\rm i}(\theta_1+\theta_3)} & -{\rm e}^{{\rm i}\theta_3} & -1 & -{\rm e}^{{\rm i}\theta_1} & 0 & 0 & 4 & 0 \\ -{\rm e}^{{\rm i}(\theta_2+\theta_3)} & -{\rm e}^{{\rm i}\theta_3} & -{\rm e}^{{\rm i}\theta_2} & -1 & 0 & 0 & 0 & 4 \end{pmatrix}. \tag{41}$$

The determinant of this matrix is exactly the same as that for the fcc lattice, i.e., $12^4 F(\theta_1, \theta_2, \theta_3)$, where $F(\theta_1, \theta_2, \theta_3)$ is given in equation (19) of [12]. Therefore,

$$z_{\rm diamond} = z_{\rm fcc}/2 \simeq 1.206\,459\,9496\ldots. \tag{42}$$

This result can be verified by taking the primitive basis with two vertices located at $(0, 0, 0)$ and $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. The corresponding matrix is given by

$$\bar{M}_{\rm diamond} = \begin{pmatrix} 4 & -1 - {\rm e}^{-{\rm i}\theta_1} - {\rm e}^{-{\rm i}\theta_2} - {\rm e}^{-{\rm i}\theta_3} \\ -1 - {\rm e}^{{\rm i}\theta_1} - {\rm e}^{{\rm i}\theta_2} - {\rm e}^{{\rm i}\theta_3} & 4 \end{pmatrix} \tag{43}$$

with determinant $12F_{\rm fcc}(\theta_1, \theta_2, \theta_3)$, where $F_{\rm fcc}(\theta_1, \theta_2, \theta_3)$ is given in equation (40).

4.4. Square lattice

Let us take the square lattice as another illustration. It has been pointed out in [11] that $d = 2$ case of the generalized body-centred-cubic lattice bcc(d) is the square lattice. The unit cell for bcc(2) with $\nu = 2$ is shown in figure 2(a). If one takes the unit cell containing two vertices as shown in figure 2(b), the corresponding matrix is

$$M_{{\rm sq},2} = \begin{pmatrix} 4 - 2\cos\theta_1 & -1 - {\rm e}^{{\rm i}\theta_2} \\ -1 - {\rm e}^{-{\rm i}\theta_2} & 4 - 2\cos\theta_1 \end{pmatrix} \tag{44}$$

with determinant

$$D_{{\rm sq},2}(\theta_1, \theta_2) = 14 - 16\cos\theta_1 + 4\cos^2\theta_1 - 2\cos\theta_2. \tag{45}$$

A possible choice of the unit cell containing three vertices for the square lattice is shown in figure 2(c). The corresponding matrix is

$$M_{{\rm sq},3} = \begin{pmatrix} 4 - 2\cos\theta_1 & -1 & -{\rm e}^{{\rm i}\theta_2} \\ -1 & 4 - 2\cos\theta_1 & -1 \\ -{\rm e}^{-{\rm i}\theta_2} & -1 & 4 - 2\cos\theta_1 \end{pmatrix} \tag{46}$$

with determinant

$$D_{{\rm sq},3}(\theta_1, \theta_2) = 52 - 90\cos\theta_1 + 48\cos^2\theta_1 - 8\cos^3\theta_1 - 2\cos\theta_2. \tag{47}$$

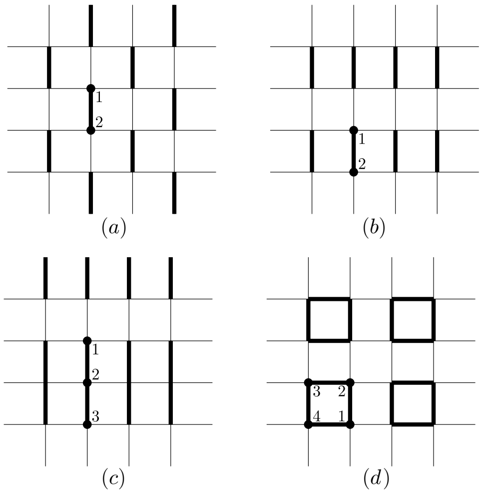

Figure 2. Different unit cells for the square lattice. (*a*) The choice for the bcc(2) lattice with $\nu = 2$. (*b*) The other choice with $\nu = 2$. (*c*) A choice with $\nu = 3$. (*d*) A choice with $\nu = 4$. Vertices within a unit cell are labelled and edges within each unit cell are connected by thick lines.

A possible choice of the unit cell containing four vertices for the square lattice is shown in figure 2(*d*). The corresponding matrix is

$$M_{\text{sq},4} = \begin{pmatrix} 4 & -1-\mathrm{e}^{-\mathrm{i}\theta_2} & 0 & -1-\mathrm{e}^{\mathrm{i}\theta_1} \\ -1-\mathrm{e}^{\mathrm{i}\theta_2} & 4 & -1-\mathrm{e}^{\mathrm{i}\theta_1} & 0 \\ 0 & -1-\mathrm{e}^{-\mathrm{i}\theta_1} & 4 & -1-\mathrm{e}^{\mathrm{i}\theta_2} \\ -1-\mathrm{e}^{-\mathrm{i}\theta_1} & 0 & -1-\mathrm{e}^{-\mathrm{i}\theta_2} & 4 \end{pmatrix} \tag{48}$$

with determinant

$$D_{\text{sq},4}(\theta_1,\theta_2) = 128 - 64(\cos\theta_1 + \cos\theta_2) + 4(\cos\theta_1 - \cos\theta_2)^2. \tag{49}$$

All these considerations give the same z_{sq} for the square lattice [6, 7],

$$\begin{aligned} z_{\text{sq}} = \frac{4C}{\pi} &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{\text{sq},2}(\theta_1,\theta_2)] \\ &= \frac{1}{3}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{\text{sq},3}(\theta_1,\theta_2)] \\ &= \frac{1}{4}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{\text{sq},4}(\theta_1,\theta_2)]. \end{aligned} \tag{50}$$

The same procedure can be carried out for lattices with different choices of unit cells, and many integral identities can be obtained.

5. Homeomorphic expansion

The method reviewed in section 2 can be applied to lattices which are not *k*-regular. We find that for a *k*-regular lattice Λ, the asymptotic growth constant of its homeomorphic expansion $\Lambda(p)$ is given by

$$z_{\Lambda(p)} = \frac{\left(\frac{\kappa}{2}-1\right)\ln(p+1) + z_{\Lambda}}{\frac{\kappa}{2}p+1}. \tag{51}$$

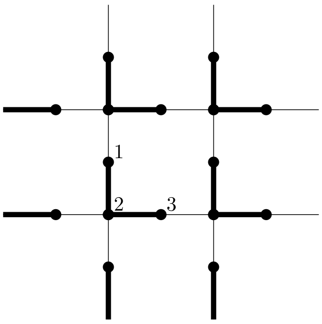

Figure 3. sq(1), the homeomorphic expansion of the square lattice with $p = 1$ vertex inserted on each edge. Vertices within a unit cell are labelled and edges within each unit cell are connected by thick lines.

This can be understood as follows. As the number of occupied edges by a spanning tree on the original lattice is $n - 1$, the number of unoccupied edges on the original lattice is $e(G) - (n-1) = (k/2)n - n + 1$. Now on each unoccupied edge we insert p degree-2 vertices. There are $p + 1$ ways to make these vertices connected in order to construct spanning trees on the homeomorphic expanded lattices, so that

$$N_{\mathrm{ST}}(\Lambda(p)) = (p+1)^{\frac{k}{2}n-n+1} N_{\mathrm{ST}}(\Lambda). \tag{52}$$

Equation (51) follows by noting that the total number of vertices of the homeomorphic expanded lattice is $(k/2)np + n$. For the homeomorphic expansion of the square, triangular and honeycomb lattices,

$$z_{\mathrm{sq}(p)} = \frac{\ln(p+1) + z_{\mathrm{sq}}}{2p+1} \qquad z_{\mathrm{tri}(p)} = \frac{2\ln(p+1) + z_{\mathrm{tri}}}{3p+1} \qquad z_{\mathrm{hc}(p)} = \frac{\frac{1}{2}\ln(p+1) + z_{\mathrm{hc}}}{\frac{3}{2}p+1}, \tag{53}$$

where p is a positive integer. For a simple illustration, consider the square lattice with one vertex inserted on each edge. As shown in figure 3, the number of vertices in each unit cell $\nu_{\mathrm{sq}(1)}$ is equal to 3. We have

$$M_{\mathrm{sq}(1)} = \begin{pmatrix} 2 & -1-\mathrm{e}^{\mathrm{i}\theta_2} & 0 \\ -1-\mathrm{e}^{-\mathrm{i}\theta_2} & 4 & -1-\mathrm{e}^{-\mathrm{i}\theta_1} \\ 0 & -1-\mathrm{e}^{\mathrm{i}\theta_1} & 2 \end{pmatrix}. \tag{54}$$

The determinant $D_{\mathrm{sq}(1)}$ is equal to $8 - 4\cos\theta_1 - 4\cos\theta_2$, which is twice the corresponding expression for the regular square lattice [11], such that $z_{\mathrm{sq}(1)} = (\ln 2 + z_{\mathrm{sq}})/3$.

6. Discussion

It is of interest to see how close the exact results presented above are to the upper bounds in equation (7). For this purpose, we define the ratio

$$r_{G_k} = \frac{z_{G_k}}{\ln b_k}, \tag{55}$$

and summarize the values of z_Λ and r_Λ for various lattices Λ in table 1. Our results agree with the observations made in [11]. Namely, z_Λ is relatively large for large value of k; while for a fixed k value, z_Λ increases with the spatial dimension d of the lattice. An example is

Table 1. Numerical values of z_Λ and r_Λ. The last digits given in the text are rounded off.

Λ	d	k	z_Λ	r_Λ
$(3^3 \cdot 4^2)$	2	5	1.406 925 832	0.948 637 1781
$(3^2 \cdot 4 \cdot 3 \cdot 4)$	2	5	1.410 855 646	0.951 286 9040
$(3^4 \cdot 6)$	2	5	1.392 023 563	0.938 589 1391
$(3 \cdot 4 \cdot 6 \cdot 4)$	2	4	1.144 801 124	0.941 142 3250
$(4 \cdot 6 \cdot 12)$	2	3	0.777 795 5061	0.929 278 9200
$c(1), nnn$	1	4	0.962 423 6501	0.791 209 5935
$c(2), nnn$	2	8	1.943 739 360	0.968 109 5375
$c(3), nnn$	3	18	2.841 787 536	0.993 304 4903
$c(4), nnn$	4	32	3.442 11	0.997 827
$c(5), nnn$	5	50	3.898 25	0.999 086
Diamond	3	4	1.206 459 950	0.991 832 1170

given by the circuit graph with additional edges connecting next-nearest-neighbour vertices, $(3 \cdot 4 \cdot 6 \cdot 4)$ lattice and the diamond structure (all with $k = 4$). The ratio r_Λ increases with dimension and approaches 1.

In summary, we have presented closed-form integrals for the asymptotic growth constants for several lattices that have not been considered before, including five Archimedean lattices, the diamond structure, the homeomorphic expansion of a regular lattice, and the d-dimensional hypercubic lattice with additional edges connecting next-nearest-neighbour vertices. In general, the asymptotic growth constant can always be expressed by an integral for a regular lattice, although closed-form evaluation may not be available. With explicit examples, we have also showed how to obtain integral identities by choosing different unit cells.

Acknowledgments

We thank Professor C-H Chen for use of computer facilities and software. S-C C thanks Professor R Shrock for helpful discussion and the collaboration on [12]. This research was partially supported by the Taiwan NSC grant NSC-94-2112-M-006-013.

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